6.S096: Number Theory Due: Friday, February 2, 2024

Problem Set 3— Fourier Analysis on Finite Abelian Groups

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Instructions: Solve any combination of problems 1-3 that sums to 90 points, as well as Problem 0 (which includes a longer survey about the course).

Collaboration is allowed (in fact, encouraged), but please list your sources and collaborators. If there are none write "**Sources consulted: none**" at the top of your solutions. Note that each student is expected to write their own solutions; it is fine to discuss the problems with others, but your writing must be your own.

The first person to report each non-trivial typo/error in any of the problem sets or lecture notes will receive 1-5 points of extra credit (depending on the severity).

Problem 1. Properties of Characters and FAFAB (50 points)

Choose and prove a total of five of the following results, such that $\max - \min \geq 6$. For example, if you do the first, then the last one has to be at least the seventh.

1. Verify that the characters on a finite abelian group G form a group, which we denote as $\hat{G}$.

2. Show that the duals (character groups) of isomorphic groups are also isomorphic.

3. Show that the characters of $\mathbb{Z}_n$ are symmetric, and so $\chi_a(b) = \chi_b(a)$. In other words, the character matrix is indeed symmetric.

4. Let G be a group of order n and $\chi \in \hat{G}$. Show that if $g \notin \ker(\chi)$ then

$$\sum_{k=0}^{n-1} \chi^k(g) = 0.$$

5. Let $G = \mathbb{Z}_m \times \mathbb{Z}_n$ and let f be a function from G to $\mathbb{C}$. Using the formula $\hat{f} = \sum_{g \in G} \langle f, B_g \rangle \delta_g$, show that

$$\hat{f}(x) = \frac{1}{\sqrt{mn}} \sum_{y \in G} \exp\left(-2\pi i \left(\frac{x_1 y_1}{m} + \frac{x_2 y_2}{n}\right)\right),$$

where $x = (x_1, x_2)$ and $y = (y_1, y_2)$.

6. Consider $f \in V_G$. Show that f is a nonzero constant function if and only if $\hat{f}$ is supported only at the identity. We say that the support of a function on G is the set of points where it is nonzero.

7. Show that $\hat{f}$ is identically zero if and only if f is identically zero.

8. Show that the averaging formula

$$\hat{f}(1) = \frac{1}{\sqrt{|G|}} \sum_{g \in G} f(g)$$

holds.

9. Prove the Fourier inversion formula: for any $f \in V_G$, we have that

$$f = \sum_{g \in G} \langle \check{f}, B_g \rangle \delta_g.$$

10. Recall $*$ to be the convolution operator (see Chapter 5). Verify that the convolution is a symmetric bilinear operator on V_G. (You may want to look up the definition of the symmetric bilinear transformation)

11. Prove that $B_s * B_t$ is nonzero exactly when $s = t$. What is its value in that case?

12. Prove that

$$\delta_a * \delta_b = \frac{1}{\sqrt{|G|}} \delta_{ab}.$$

13. Show that $\langle f, g * h \rangle = \langle g, f * h \rangle$.

Problem 2. Computing Characters (40 points)

Let G be a finite group (not necessarily abelian).

1. We showed in class that if G is abelian, then $G \cong \hat{G}$. Where did we use the abelian part?

2. We say that two elements g, h of a group G are conjugate if $g = xhx^{-1}$ for some $x \in G$. Show that this is an equivalence relation (reflexive, symmetric, and transitive).

3. Show that a character is constant (takes only one value) on a conjugacy class.

4. Implement a program to compute the Fourier transform of a signal (which is inputted as an array of complex numbers) using the formula. What is the complexity of your algorithm?

5. Compare your runtime against that of a built-in library (such as numpy) for some randomly selected signals of length n, where $n = 10, 100, 1000, 10000$. Plot this for different values of n and use this to guesstimate what the runtime of both algorithms is, in terms of the length of the array.

Problem 3. Previous Problem (40 points)

Solve one of problems 2-4 which you did not solve on the previous pset.

Problem 0. Survey (10 points)

Complete the following survey by rating each of the problems you solved on a scale of 1 to 10 according to how interesting you found the problem (1 = "mind numbing," 10 = "mind blowing"), and how difficult you found the problem (1 = "trivial," 10 = "brutal"). Also estimate the amount of time you spent on each problem to the nearest half hour. [1]

	Interest	Difficulty	Time Spent
Problem 1			
Problem 2			
Problem 3			
Problem 4			
Problem 5			
Problem 6			

Please rate each of the following lectures/sessions that you attended on a scale of 1 to 10, according to the quality of the material (1="pointless", 10="priceless"), the quality of the presentation (1="epic fail", 10="perfection"), the pace (1="watching paint dry", 10="head still spinning"), and the novelty of the material (1="old hat", 10="all new").

Did you attend lectures in-person on the last week? If so, which ones, so we could grant you extra credit.

Date	Lecture Topic	Material	Presentation	Pace	Novelty
01/23	Signal Processing & FAFAB				
01/24	FAFAB and Analytic Number Theory				
01/25	Bonus				

Feel free to record any additional comments you have on the problem sets or lectures; in particular, how you think they could be improved (which they surely can!).

We would also appreciate general feedback about the entire class. Since this is the first time we are offering the class, there are many things we can improve on, and we would appreciate your input.

[1] This survey, as well the template of the Pset, is copied from Andrew Sutherland's 18.783 problem sets.